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(1) *Constant rule:* If c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

eg. $\frac{d}{dx}[3] = 0$, $\frac{d}{dx}[\sqrt{2}] = 0$, $\frac{d}{dx}[3^\pi] = 0$.

(2) *Power rule:* If n is a real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

eg. $\frac{d}{dx}[x^4] = 4x^3$, $\frac{d}{dx}[x^{-2}] = -2x^{-3}$, $\frac{d}{dx}\left[\sqrt[7]{x^3}\right] = \frac{d}{dx}\left[x^{\frac{3}{7}}\right] = \frac{3}{7}x^{-\frac{4}{7}}$.

(3) *Constant multiple rule:* If f is a differentiable function of x and c is a real number, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x).$$

eg. $\frac{d}{dx}[2x^5] = 2\frac{d}{dx}[x^5] = 10x^4$, $\frac{d}{dx}\left[\sqrt{3}x^{\frac{5}{2}}\right] = \sqrt{3}\frac{d}{dx}\left[x^{\frac{5}{2}}\right] = \sqrt{3}\frac{5}{2}x^{\frac{3}{2}} = \frac{5\sqrt{3}}{2}x^{\frac{3}{2}}$.

Remark: Note that derivative of a constant function is zero (constant rule). But derivative of a function multiplied by a constant may not be zero (constant multiple rule).

(4) *Sum and difference rules:* If f and g are differentiable functions of x, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$
$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$$

eg. $\frac{d}{dx}[x^2 + x] = \frac{d}{dx}[x^2] + \frac{d}{dx}[x] = 2x + 1$.

Special application: Suppose f and g are differentiable functions of x, and $f(x) = g(x) + 2x^5 - 1$. If $f'(1) = -2$ find $g'(1)$.
Solution: Differentiating both sides of $f(x) = g(x) + 2x^5 - 1$ with respect to x, we have

$$f'(x) = g'(x) + 10x^4.$$

Putting $x = 1$ in both sides we have

$$f'(1) = g'(1) + 10.$$

We know that $f'(1) = -2$, therefore $g'(1) = -12$.

(5) *Product rule:* If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}(f(x))\right]g(x) + f(x)\frac{d}{dx}[g(x)] = f'(x)g(x) + f(x)g'(x).$$

eg.

$$\begin{aligned}
\frac{d}{dx}[(x^2+1)(x^3+x)] &= \left[\frac{d}{dx}(x^2+1)\right](x^3+x) + (x^2+1)\frac{d}{dx}[x^3+x] \\
&= [2x+0](x^3+x) + (x^2+1)[3x^2+1] \\
&= 2x(x^3+x) + (x^2+1)(3x^2+1).
\end{aligned}$$

General product rule: If we have product of three or more functions, then product rule is

$$\begin{aligned}
\frac{d}{dx}[f_1(x)f_2(x)f_3(x)] &= f_1'(x)f_2(x)f_3(x) + f_1(x)f_2'(x)f_3(x) + f_1(x)f_2(x)f_3'(x) \\
\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)] &= f_1'(x)f_2(x)f_3(x)f_4(x) + f_1(x)f_2'(x)f_3(x)f_4(x) \\
&\quad + f_1(x)f_2(x)f_3'(x)f_4(x) + f_1(x)f_2(x)f_3(x)f_4'(x)
\end{aligned}$$

and so on.

(6) *Quotient rule:* If f and g are differentiable functions and $\underline{g(x) \neq 0}$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

eg.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2+3x}{x+2}\right] &= \frac{(x+2)\frac{d}{dx}[x^2+3x] - (x^2+3x)\frac{d}{dx}[x+2]}{(x+2)^2} \\
&= \frac{(x+2)(2x+3) - (x^2+3x)(1+0)}{(x+2)^2} \\
&= \frac{(x+2)(2x+3) - (x^2+3x)}{(x+2)^2}.
\end{aligned}$$

(7) *Chain rule:* If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x then $y = f(g(x))$ is also a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Or, equivalently

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

eg.

1. Take $y = \sin(x^2+1)$. Notice that if we take $y = f(u) = \sin u$ and $u = g(x) = x^2+1$, then we have $y = f(g(x)) = \sin(x^2+1)$. By chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\sin u] \frac{d}{dx}[x^2+1] \\ &= [\cos u](2x+0) \\ &= 2x\cos(x^2+1), \quad (\text{since } u = x^2+1). \end{aligned}$$

2. $y = \tan(\ln x^2)$. Applying chain rule repeatedly we have

$$\begin{aligned} \frac{dy}{dx} &= \sec^2(\ln x^2) \frac{d}{dx}[\ln x^2] \\ &= \sec^2(\ln x^2) \frac{1}{x^2} \frac{d}{dx}[x^2] \\ &= \sec^2(\ln x^2) \frac{1}{x^2} \cdot 2x \\ &= \frac{2}{x} \sec^2(\ln x^2). \end{aligned}$$

Remark: At the first step, I considered $(\ln x^2)$ as x. We know that $\frac{d}{dx} \tan x = \sec^2 x$. That's why I got $\sec^2(\ln x^2)$ at the first step. But $(\ln x^2)$ is not x. So according to chain rule (equivalent version), we have to multiply by $\frac{d}{dx}[\ln x^2]$. Now I consider x^2 as x. We know that $\frac{d}{dx}[\ln x] = \frac{1}{x}$, that's why I got $\frac{1}{x^2}$. But x^2 is not x. So again we have to apply chain rule and multiply the whole thing by $\frac{d}{dx}[x^2]$. Now we know that $\frac{d}{dx}[x^2] = 2x$, and we are done!

(8) *General power rule:* If $f(x)$ is a differentiable function of x. Then for any real number n we have

$$\frac{d}{dx}[(f(x))^n] = n(f(x))^{n-1} \cdot f'(x).$$

This rule is a consequence of chain rule.

eg. $y = (\sin x)^3$. Taking $f(x) = \sin x$ and using the general power rule we have

$$\begin{aligned} \frac{dy}{dx} &= 3(\sin x)^2 \cdot \frac{d}{dx}[\sin x] \\ &= 3(\sin x)^2 \cos x. \end{aligned}$$